2024

calculus

Differential methods

Real number

Real numbers are a fundamental concept in mathematics. They include all rational and irrational numbers and can be represented on the number line. Here's a breakdown:

1. **Rational Numbers**: These are numbers that can be expressed as a fraction of two integers, where the denominator is not zero. Examples include $\frac{1}{2}$,- $\frac{8}{5}$, , etc.

2. **Irrational Numbers**: These are numbers that cannot be expressed as a fraction of two integers. They have non-repeating, non-terminating decimal expansions. Examples include 22, π, e, etc.

3. **Integers**: These are whole numbers and their negatives, including zero. Examples include -3, -2, -1, 0, 1, 2, 3, etc.

4. **Whole Numbers**: These are non-negative integers, including zero. Examples include 0, 1, 2, 3, etc.

5. **Natural Numbers**: These are the counting numbers, starting from 1. Examples include 1, 2, 3, 4, etc.

Intervals

In mathematics, an interval is a set of real numbers with the property that any number that lies between two numbers in the set is also included in the set. Intervals are commonly used in various mathematical contexts, including calculus, analysis, and geometry. There are several types of intervals, each with its own notation and characteristics:

1. **Open Interval**: An open interval does not include its endpoints. It is denoted using round parentheses. For example, the open interval from 2 to 5 is written as (2,5)(2,5), and it includes all real numbers greater than 2 and less than 5.

2. **Closed Interval**: A closed interval includes both of its endpoints. It is denoted using square brackets. For example, the closed interval

from -3 to 1 is written as [−3,1][−3,1], and it includes all real numbers greater than or equal to -3 and less than or equal to 1.

3. **Half-Open or Half-Closed Interval**: These intervals include one endpoint but not the other. There are two types:
   - Half-open intervals include one endpoint and not the other. They are denoted with one parenthesis and one square bracket. For example, $(a,b]$ includes all real numbers greater than a and less than or equal to b.
   - Half-closed intervals include one endpoint and not the other. They are denoted with one square bracket and one parenthesis. For example, $[a,b)$ includes all real numbers greater than or equal to a and less than b.

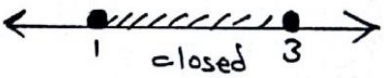
الفترات (Intervals) :

هي جزء من خط الأعداد الحقيقية

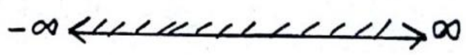
مثال : الاعداد الحقيقية بين 1 و 3 هي فترة (interval)

اذا لم يكن العددان 1 و 3 في الفترة تكون مفتوحة ونكتبها بهذا الشكل $(1, 3)$

اذا كان العددان 1 و 3 في الفترة تكون مغلقة ونكتبها بهذا الشكل $[1, 3]$



$R = (-\infty, \infty)$ مجموعة الأعداد الحقيقية مفتوحة



Domain of function

The domain in mathematics refers to the set of possible values for the independent variable (often denoted as x) that make the function defined and yield meaningful results.

In other words, if you have a function $f(x)$, the domain is the set of all values that the variable x can take on such that $f(x)$ is defined.

Let's take an example:

If we have the function $f(x)=\sqrt{x}$, then the domain is the set of non-negative real numbers ($x \geq 0$), because the square root cannot take a negative value in the real numbers.

Ex: find domain:

① $f(x)=\sqrt{9-x^2}$

sol: $0 \leq 9-x^2$

$9-x^2=(3-x)(3+x)=0$

$3-x=0 \qquad 3+x=0$

$3=x \qquad x=-3$

$-- \quad \bullet_{-3} \quad ++ \checkmark \quad \bullet_{3} \quad --$

$D_f=[-3,3]$

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② $f(x)=\sqrt{x^2-4x+3}$

sol: $0 \leq x^2-4x+3$

$x^2-4x+3=(x-1)(x-3)=0$

$x=1, \ x=3$

$-\infty \quad ++ \checkmark \quad \bullet_{1} \quad -- \quad \bullet_{3} \quad ++ \checkmark \quad \infty$

$D_f=(-\infty,1] \cup [3,\infty)$

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③ $f(x)=\ln(2x-x^2)$

sol: $0 < 2x-x^2$

$-- \quad \circ_{0} \quad ++ \checkmark \quad \circ_{2} \quad --$

$2x-x^2=x(2-x)=0$

$x=0, \ x=2$

$D_f=(0,2)$

Composite Function

Let f (χ) and g(χ) be two functions

We define: $fog\ (x)=f(g\ x\ )$

Ex: Let $f\ x = x2$, $g\ x = x-7$ evaluate $fog\ \ x$ and $(gof)(x)$

Sol: $fog(x) = f(g(x)) = f(x-7) = (x-7)^2$

$gof(x) = g(f(x)) = g(x^2) = x^2 - 7$

$\therefore fog \neq (gof)$

Ex: $f(x) = x^2+1$, $g(x) = \sqrt{x-2}$

Find: ① $fog(4)$ ② $fog(x)$ ③ $gof(x)$.

Sol: ① $fog(4) = f(g(4))$

$= f(\sqrt{2})$

$= (\sqrt{2})^2 + 1 = 2+1 = 3$

② $fog = f(g) = g^2+1 = (\sqrt{x-2})^2 + 1$

$= x-2+1$

$= x-1$

③ $gof = g(f) = \sqrt{f-2} = \sqrt{x^2+1-2}$

$= \sqrt{x^2-1}$

Operation on function

If $f(x)$, $g(x)$ two functions :

(*) $(f \pm g)(x) = f(x) \pm g(x)$

$(f.g)(x) = f(x)\, g(x)$

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$

(*) Dom $(f \pm g)$ = Dom f $\cap$ Dom g

Dom $(\frac{f}{g})$ = (Dom f $\cap$ Dom g) $-$ $\{g=0\}$

Ex| $f(x) = 2\sqrt{x}$, $g(x) = \sqrt{x}$

find: ① $f+g$, $f-g$, $f.g$ and it's domain.

sol: $(f+g)(x) = 2\sqrt{x} + \sqrt{x} = 3\sqrt{x}$

$(f-g)(x) = 2\sqrt{x} - \sqrt{x} = \sqrt{x}$

$(f.g)(x) = (2\sqrt{x})(\sqrt{x}) = 2(\sqrt{x})^2 = 2x$

Dom $(f \pm g)$ = Dom f $\cap$ Dom g

$= [0,\infty) \cap [0,\infty) = [0,\infty)$

② $\frac{f}{g}$ and it's domain.

sol: $\frac{f}{g}(x) = \frac{2\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}} = 2$

Dom $(\frac{f}{g})$ = (Dom f $\cap$ Dom g) $-$ $\{g=0\}$

$= [0,\infty) - \{0\}$ $\qquad \sqrt{x}^2 = 0^2$

$= (0,\infty)$ $\qquad x = 0$

Ex| $f(x) = \sqrt{x-2}$, $g(x) = x-4$

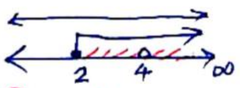
find domain: ① $\frac{f}{g}$ ② $\frac{g}{f}$

sol: ① Dom $(\frac{f}{g})$ = (Dom f $\cap$ Dom g) $-$ $\{g=0\}$

$0 \le x-2$

$2 \le x$

$= ([2,\infty) \cap \mathbb{R}) - \{x-4=0\}$



$= [2,\infty) - \{4\}$ ✓

$= [2,4) \cup (4,\infty)$

$A \cap \mathbb{R} = A$

$A \cap B = B \cap A$

② Dom $(\frac{g}{f})$ = (Dom g $\cap$ Dom f) $-$ $\{f=0\}$

$= [2,\infty) - \{\sqrt{x-2}^2 = 0^2\}$

$= [2,\infty) - \{2\}$

$= (2,\infty)$

Inverse Function

Given a function F with domain A and the range B.

The inverse function of f written f, is a function with domain B and range A such that for every y ∈ B there exists only x ∈ A with $x = f^{-1}(y)$ Note that: $f^{-1} \neq \frac{1}{f}$

Remark: $f(f^{-1}(x)) = x$ نستخدم لايجاد $f^{-1}(x)$

Ex Find $f^{-1}(x)$:

1. $f(x) = x^5 - 2$

Sol: $f(f^{-1}(x)) = x$

$(f^{-1}(x))^5 - 2 = x$

$\sqrt[5]{(f^{-1}(x))^5} = \sqrt[5]{x+2}$

$f^{-1}(x) = \sqrt[5]{x+2}$

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2. $f(x) = \sqrt[3]{x+4}$

Sol: $f(f^{-1}(x)) = x$

$\left(\sqrt[3]{f^{-1}(x) + 4}\right)^3 = (x)^3$

$f^{-1}(x) + 4 = x^3$

$f^{-1}(x) = x^3 - 4$

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3. $f(x) = \frac{2x+1}{3}$ (H·W#) $\frac{3x-1}{2}$ ✓

4. $f(x) = \frac{x+1}{2-x}$ (H·W#)

logarithm

A logarithm is a mathematical function that converts numbers into other numbers based on a specific relationship with a given base. The natural logarithm is the logarithm that uses the number *e* (which is approximately 2.71828) as its base. Typically, the natural logarithm is denoted by ln(*x*), where *x* is the number for which the natural logarithm is calculated.

The logarithm to base *a* of a number *x* is usually written as $\log_a x$ In this context, *a* is the base, and *x* is the number for which the logarithm is calculated.

$$f(x) = \mathrm{Log}_b(x) \quad , \quad b>0,\ b \neq 1$$

$$(*)\ \mathrm{Log}_{10} x = \log x \quad , \quad \mathrm{Log}_e x = \ln x$$

$$\cancel{\mathrm{Log}_b}\, \cancel{b}^{x} = x$$

$$\mathrm{Log}_b b = \log_b \cancel{b}^{1} = 1 \qquad \mathrm{Log}_b 1 = 0$$

$$\mathrm{Log}_b 1 = \log_b \cancel{b}^{0} = 0 \qquad \mathrm{Log}_b b = 1$$

Ex Evaluate:

① $\log_2 8 = \cancel{\log_2} 2^3 = 3$

$$\begin{array}{c|c} 2 & 8 \\ 2 & 4 \\ 2 & 2 \\ & 1 \end{array}$$

② $\log_4 \frac{1}{16} = \cancel{\log_4} 4^{-2} = -2$

$\frac{1}{16} = \frac{1}{4^2} = 4^{-2}$

③ $\log_7 1 = \cancel{\log_7} 7^0 = 0$

④ $\ln e = \log_e e = \cancel{\log_e} e^1 = 1$

⑤ $\log(0.001) = \cancel{\log_{10}} 10^{-3} = -3$

$0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$

⑥ $\log_{27} 9 = \frac{2}{3}$ H.W#

Remark: $\text{Dom}(b^x) = \mathbb{R}$

$\text{Dom} \log_b(f(x))$: $0 < f(x)$

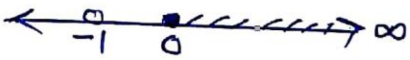
⑦ $\log_3(-3)$ undefine ⑧ $\ln(0)$ undefine

Ex find domain:

① $f(x) = e^{\sqrt{x}} + \frac{1}{x+1}$

sol: $0 \leq x$ and $x \neq -1$

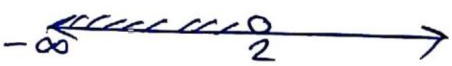
$\text{Dom} f = [0, \infty)$



② $f(x) = \log_5(2-x)$

sol: $0 < 2-x$

$x < 2$



$\text{Dom} f = (-\infty, 2)$

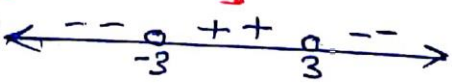
③ $f(x) = \ln(9-x^2)$

sol: $0 < 9-x^2$

$9-x^2=0$

$9=x^2$

$\pm 3 = x$



$\text{Dom} f = (-3, 3)$

④ $f(x) = \ln(\sqrt{x^2-4})$

⑤ $f(x) = \frac{1}{\ln(x-3)}$

⑥ $f(x) = \frac{e^x + x^3}{\log_3(x+1) - 2}$

H.W#